Helical harmonics for static fields

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Helical harmonic solutions of Laplace's equation are derived using a right-handed nonorthogonal helical coordinate system and a modified separation of variables technique. Unlike the Cartesian or cylindrical coordinate systems, in the helical system, two different sets of solutions are admitted, one right handed and one left handed. Consequently, the helical harmonics are used to solve the boundary value problem of two nested helices with different radii set to different potential values.

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I. INTRODUCTION

Often it is believed that orthogonal coordinate systems are the only coordinate systems that can be usefully applied to various boundary-value problems in physics. Although it is realized that nonorthogonal systems composed of coordinates coinciding with the boundary surfaces of a given geometry are useful for straightforward application of boundary conditions, it is known also that the partial differential equations (PDE's) associated with such coordinates are (in almost all cases) extremely difficult to solve analytically. In particular, it is well known that the powerful separation of variables technique will not work in general on PDE's obtained using nonorthogonal coordinates.

However, it is possible occasionally to find a nonorthogonal system whose associated PDE's can be solved using separation of variables. Such a system is a right-handed nonorthogonal helical coordinate system denoted by (ρ, ϕ, ζ) [1]. This system is related to the Cartesian system via the transformation [2–3]

$$x=\rho\cos\phi, \tag{1a}$$

$$y=\rho\sin\phi, \tag{1b}$$

$$z=\zeta+\bar{p}\phi, \tag{1c}$$

where $\bar{p}=p/2\pi>0$, p is the helical pitch, and the pitch angle α_0 is defined as

$$\tan\alpha_0=\frac{\bar{p}}{\rho}, \tag{2}$$

(see Fig. 1).

This helical system is closely related to the standard cylindrical coordinate system (r,θ,z) by the coordinate relationships

$$r=\rho, \tag{3a}$$

$$|\theta|=|\phi|, \tag{3b}$$

$$z=\zeta+\bar{p}\phi, \tag{3c}$$

but where θ and ϕ are measured along different curves. The cylindrical/helical transformation of unit vectors is given by [2]

$$\hat{\rho}=\hat{r}, \tag{4a}$$

$$\hat{\phi}=\cos\alpha_0\hat{\theta}+\sin\alpha_0\hat{z}, \tag{4b}$$

$$\hat{\zeta}=\hat{z}. \tag{4c}$$

In the limit, as the pitch (or the pitch angle) goes to zero, the helical system reduces to the standard cylindrical system. Although $\hat{\rho}\cdot\hat{\zeta}=0$ and $\hat{\rho}\cdot\hat{\phi}=0$, the nonorthogonality of this system occurs because $\hat{\phi}\cdot\hat{\zeta}=\sin\alpha_0$, which is nonzero in general.

In the following, we solve Laplace's equation in the above nonorthogonal helical coordinate system using a somewhat modified separation of variables technique. The resulting analytic solutions are referred to as helical harmonic functions or simply *helical harmonics*. Unlike the Cartesian or cylindrical coordinate systems, in the helical system, two different types of solutions are admitted, one right-handed and one left-handed.

In Sec. II Laplace's equation is solved generally in the helical coordinate system. In Sec. III, some unusual properties of the helical radial functions are discussed. In Sec. IV the right-handed and left-handed types of solutions are discussed. In Sec. V, the helical harmonics are used to solve the

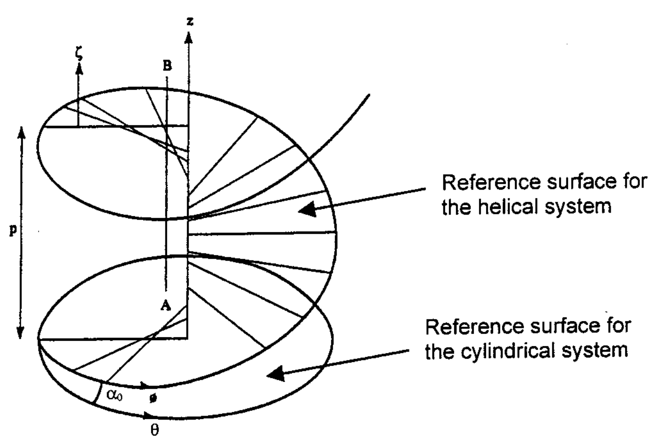

FIG. 1. A nonorthogonal helical coordinate system [1].

boundary-value problem of two nested helices of different potentials with different radii. Section VI contains the conclusions.

II. LAPLACE'S EQUATION IN HELICAL COORDINATES

Laplace's equation in the helical coordinate system of Sec. I is

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial\zeta^2}+\frac{1}{\rho^2}\left(\frac{\partial^2}{\partial\phi^2}+\bar{p}^2\frac{\partial^2}{\partial\zeta^2}-2\bar{p}\frac{\partial^2}{\partial\zeta\partial\phi}\right)\right]\Phi(\rho,\phi,\zeta)=0. \tag{5}$$

Allowing

$$\Phi(\rho,\phi,\zeta)=R(\rho)P(\phi)Z(\zeta), \tag{6}$$

by substituting Eq. (6) into Eq. (5), Eq. (5) becomes

$$\frac{R''(\rho)}{R(\rho)}+\frac{R'(\rho)}{\rho R(\rho)}+\frac{P''(\phi)}{\rho^2 P(\phi)}+\left(1+\frac{\bar{p}^2}{\rho^2}\right)\frac{Z''(\zeta)}{Z(\zeta)}-\frac{2\bar{p}}{\rho^2}\frac{P'(\phi)Z'(\zeta)}{P(\phi)Z(\zeta)}=0. \tag{7}$$

The primes in Eq. (7) denote differentiation with respect to ρ, ϕ, and ζ as appropriate. Although the variables in Eq. (7) cannot be separated in the usual way, if P''/P, Z''/Z, and $P'Z'/PZ$ are all constant, Eq. (7) can be reduced to an ordinary differential equation (ODE) in the variable ρ alone. The above quantities can all be constant when $Z(\zeta)$ and $P(\phi)$ are assumed *a priori* to be exponential functions.

Let

$$P(\phi)=e^{\pm\Omega\phi};\quad Z(\zeta)=e^{\pm\gamma\zeta}, \tag{8}$$

where

$$\Omega=\nu+i\mu \tag{9a}$$

and

$$\gamma=\alpha+i\beta. \tag{9b}$$

Ω and γ are separation constants (not functions of the coordinates) and are complex. The constants, μ, ν, α, and β are all assumed to be real and positive. Upon taking the appropriate derivatives of Eq. (8) and substituting them into Eq. (7), Eq. (7) becomes

$$\frac{R''}{R}+\frac{R'}{\rho R}+\gamma^2+\frac{1}{\rho^2}[\Omega^2+(\bar{p}\gamma)^2-2\bar{p}(\pm\Omega)(\pm\gamma)]=0. \tag{10}$$

Because of the mixed partial term in Eq. (5), Eq. (10) is actually two different equations. If the signs of the exponential terms in Eq. (8) are the same, either both positive or both negative, Eq. (10) can be written as

$$R''+\frac{R'}{\rho}+\gamma^2 R+\frac{1}{\rho^2}(\Omega-\bar{p}\gamma)^2 R=0. \tag{11}$$

If the signs of the exponential terms in Eq. (8) are different, Eq. (10) becomes

$$R''+\frac{R'}{\rho}+\gamma^2 R+\frac{1}{\rho^2}(\Omega+\bar{p}\gamma)^2 R=0. \tag{12}$$

Equations (11) and (12) are both forms of Bessel's equation. The general solution of Eq. (11) is

$$R(\rho)=a_1 J_{i(\Omega-\bar{p}\gamma)}(\gamma\rho)+b_1 J_{-i(\Omega-\bar{p}\gamma)}(\gamma\rho), \tag{13}$$

while the solution of Eq. (12) is

$$R(\rho)=a_2 J_{i(\Omega+\bar{p}\gamma)}(\gamma\rho)+b_2 J_{-i(\Omega+\bar{p}\gamma)}(\gamma\rho). \tag{14}$$

The radial functions given by Eqs. (13) and (14) consist of Bessel functions of both complex argument and complex order. Thus two general solutions of Eq. (5) for the potential in the helical coordinate system result. One is

$$\Phi(\rho,\phi,\zeta)=[a_1 J_{i(\Omega-\bar{p}\gamma)}(\gamma\rho)+b_1 J_{-i(\Omega-\bar{p}\gamma)}(\gamma\rho)]\times[c_1 e^{\Omega\phi}e^{\gamma\zeta}+d_1 e^{-\Omega\phi}e^{-\gamma\zeta}]. \tag{15}$$

The other is

$$\Phi(\rho,\phi,\zeta)=[a_2 J_{i(\Omega+\bar{p}\gamma)}(\gamma\rho)+b_2 J_{-i(\Omega+\bar{p}\gamma)}(\gamma\rho)]\times[c_2 e^{\Omega\phi}e^{-\gamma\zeta}+d_2 e^{-\Omega\phi}e^{\gamma\zeta}]. \tag{16}$$

Equations (15) and (16) are two analytic exact solutions of Laplace's equation in the right-handed nonorthogonal helical coordinate system of Sec. I. They are the helical harmonics. The Bessel functions occurring in the radial portion of Eqs. (15) and (16) will be referred to throughout as the *helical Bessel functions*.

Formally, both Eqs. (15) and (16) are similar to the cylindrical harmonics [4–5] but with several important differences. First, in both Eqs. (15) and (16), the parameter $\Omega=\nu+i\mu$ does not necessarily have to be an integer as it would in a standard cylindrical system. Second, since $\bar{p}\gamma$ is not (in general) an integer quantity either, the Bessel function orders in Eqs. (15) and (16) are noninteger in general and may be complex. Third, the Bessel functions in Eqs. (15) and (16) have orders that are functions of the separation constant, γ, along the ζ direction and are no longer constant for different values of γ. For both Eqs. (15) and (16) in the limit as the normalized pitch $\bar{p}$ goes to zero, these equations reduce to the same solution that is the usual cylindrical harmonic solution where $\Omega=\pm i\mu$ and $\gamma=\pm i\beta$ with μ restricted to being an integer.

Note that a helically symmetric solution, i.e., no ϕ dependence, occurs when the separation constant Ω is zero. For this circumstance, Eqs. (15) and (16) again reduce to the same solution given by

$$\Phi=[aJ_{i\bar{p}\gamma}(\gamma\rho)+bJ_{-i\bar{p}\gamma}(\gamma\rho)][ce^{\gamma\zeta}+de^{-\gamma\zeta}]. \quad (17)$$

Two special cases of Eqs. (15) and (16) will be considered.

A. Case I: $\nu=\alpha=0$

When the separation constants, Ω and γ, are purely imaginary, we are assuming no attenuation or gain in either the ϕ or ζ directions. In this case, Eq. (15) becomes

$$\Phi=[a_1'I_{\mu-\bar{p}\beta}(\beta\rho)+b_1'I_{-(\mu-\bar{p}\beta)}(\beta\rho)]\times[c_1'e^{i\mu\phi}e^{i\beta\zeta}+d_1'e^{-i\mu\phi}e^{-i\beta\zeta}], \quad (18)$$

while Eq. (16) becomes

$$\Phi=[a_2'I_{\mu+\bar{p}\beta}(\beta\rho)+b_2'I_{-(\mu+\bar{p}\beta)}(\beta\rho)]\times[c_2'e^{i\mu\phi}e^{-i\beta\zeta}+d_2'e^{-i\mu\phi}e^{i\beta\zeta}]. \quad (19)$$

In Eqs. (18) and (19), the radial functions are now modified Bessel functions with arguments and orders that are purely real. In Eq. (18) if $c_1'=d_1'=1$ or $c_1'=-d_1'=1$, Eq. (18) could have been written as

$$\Phi=[a_1'I_{\mu-\bar{p}\beta}(\beta\rho)+b_1'I_{-(\mu-\bar{p}\beta)}(\beta\rho)]\begin{Bmatrix}\cos(\mu\phi+\beta\zeta)\\ \sin(\mu\phi+\beta\zeta)\end{Bmatrix}, \quad (20)$$

where the factors of $\frac{1}{2}$ or $\frac{1}{2i}$ have been included in a_1' and b_1'. Similarly, in Eq. (19) if $c_2'=d_2'=1$ or $c_2'=-d_2'=1$, Eq. (19) could have the form

$$\Phi=[a_2'I_{\mu+\bar{p}\beta}(\beta\rho)+b_2'I_{-(\mu+\bar{p}\beta)}(\beta\rho)]\begin{Bmatrix}\cos(\mu\phi-\beta\zeta)\\ \sin(\mu\phi-\beta\zeta)\end{Bmatrix}. \quad (21)$$

Note that Eqs. (20) and (21) exhibit a fundamental difference between the helical and cylindrical coordinate systems. In the helical system, unless $\mu=0$, it is not possible to have a completely separable solution with the trigonometric form

$$\Phi_h=I_{(\mu-\bar{p}\beta)}(\beta\rho)\cos\mu\phi\cos\beta\zeta, \quad (22)$$

while in a cylindrical system, the separable form

$$\Phi_c=I_n(\beta r)\cos n\theta\cos\beta z, \quad (23)$$

(where n is now an integer) is always possible.

B. Case II: $\nu=\beta=0$

When the separation constant along the ϕ direction is purely imaginary, but the separation constant along the ζ direction is purely real, there is attenuation (or gain) along ζ. Thus, Eq. (15) becomes

$$\Phi=[a_1'J_{(\mu+i\bar{p}\alpha)}(\alpha\rho)+b_1'J_{-(\mu+i\bar{p}\alpha)}(\alpha\rho)][c_1'e^{i\mu\phi}e^{\alpha\zeta}+d_1'e^{-i\mu\phi}e^{-\alpha\zeta}], \quad (24)$$

while Eq. (16) becomes

$$\Phi=[a_2'J_{(\mu-i\bar{p}\alpha)}(\alpha\rho)+b_2'J_{-(\mu-i\bar{p}\alpha)}(\alpha\rho)]\times[c_2'e^{i\mu\phi}e^{-\alpha\zeta}+d_2'e^{-i\mu\phi}e^{\alpha\zeta}]. \quad (25)$$

(The values of the constants used in Eqs. (24) and (25) are not the same as those in Eqs. (20) and (21).)

In Eqs. (24) and (25), the helical Bessel functions have real arguments but complex orders. Regardless of the fact that μ, α, ρ, and ζ are real and positive, the helical Bessel functions in this case are complex.

The other special cases, where $\mu=\alpha=0$ and $\mu=\beta=0$ are, of course, similarly done.

III. SOME PROPERTIES OF THE HELICAL BESSEL FUNCTIONS

It is important to realize that because the separation constant in the ζ direction shows up in the order, as well as the argument, of the helical Bessel functions, the behavior of these functions is somewhat different from that of the cylindrical Bessel functions where the order is a fixed integer with the order independent of the parameters appearing in the argument.

The two special cases in Sec. II will be considered in detail. Assuming that $\nu=\alpha=0$ as in Case I, the potential is given by Eqs. (18) and (19). By assuming that $\mu=0$ also, the helically symmetric solution for this case is given by

$$\Phi=[a_1I_{\bar{p}\beta}(\beta\rho)+b_1I_{-\bar{p}\beta}(\beta\rho)][c_1e^{i\beta\zeta}+d_1e^{-i\beta\zeta}]. \quad (26)$$

In Eq. (26), the parameters $\bar{p}$, β, ρ, and ζ are all positive and real. When $\bar{p}\beta$ is noninteger, $I_{\bar{p}\beta}(\beta\rho)$ and $I_{-\bar{p}\beta}(\beta\rho)$ are linearly independent. Under the condition that $\bar{p}\beta$ is integer, Eq. (26) must take the form

$$\Phi=[aI_{\bar{p}\beta}(\beta\rho)+bK_{\bar{p}\beta}(\beta\rho)][ce^{i\beta\zeta}+de^{-i\beta\zeta}]. \quad (27)$$

Note that as β tends to zero for $\bar{p}$, ρ fixed,

$$\lim_{\beta\to0}I_{\bar{p}\beta}(\beta\rho)=1 \quad (28a)$$

and

$$\lim_{\beta\to0}I_{-\bar{p}\beta}(\beta\rho)=1 \quad (28b)$$

also, but

$$\lim_{\beta\to0}K_{\bar{p}\beta}(\beta\rho)=+\infty. \quad (28c)$$

For $\bar{p}$ and β fixed, as the radial coordinate, ρ, approaches zero

$$\lim_{\rho\to0}I_{\bar{p}\beta}(\beta\rho)=0 \quad (29a)$$

but

$$\lim_{\rho\to0}I_{-\bar{p}\beta}(\beta\rho)=+\infty, \quad (29b)$$

while

$$\lim_{\rho\to0}K_{\bar{p}\beta}(\beta\rho)=+\infty. \quad (29c)$$

For $K_{\bar{p}\beta}(\beta\rho)$, the approach to infinity as $\rho\to 0$ is a strong function of the order, $\bar{p}\beta$. For $I_{\bar{p}\beta}(\beta\rho)$ and $I_{-\bar{p}\beta}(\beta\rho)$, when $\bar{p}\beta$ is integer, these two functions are identical, but they are also very close to each other in value whenever $\beta\rho$ exceeds $|\bar{p}\beta|$. Both these functions increase without bound as β or ρ becomes large.

Thus, for any region of space where the radial coordinate is close to zero, one should use the $I_{\bar{p}\beta}(\beta\rho)$ function since it is finite there. When the radial coordinate is large, the $K_{\bar{p}\beta}(\beta\rho)$ function should be used since it tends to zero as β or ρ becomes large.

Whether one uses $I_{\bar{p}\beta}(\beta\rho)$, $I_{-\bar{p}\beta}(\beta\rho)$, or $K_{\bar{p}\beta}(\beta\rho)$, none of these functions possess real zeros when $\bar{p}$, β, and ρ are positive and real. Thus, the solution in Eq. (27) can only be used for boundary-value problems in which zeros occur along the ζ direction, i.e., along the helix, not the radial direction.

The above modified Bessel functions can be written in a somewhat different form. If one assumes that

$$\bar{p}\beta=\sigma, \tag{30}$$

where σ is a dimensionless quantity, then using $\beta=\sigma/\bar{p}$, one can write

$$I_{\bar{p}\beta}(\beta\rho)=I_{\sigma}(\sigma\rho/\bar{p}). \tag{31}$$

Using $\rho/\bar{p}=ctn\ \ \alpha_0=\chi$, another dimensionless quantity, where α_0 is the pitch angle, Eq. (31) becomes

$$I_{\bar{p}\beta}(\beta\rho)=I_{\sigma}(\sigma\chi). \tag{32}$$

Both $I_{\sigma}(\sigma\chi)$ and $K_{\sigma}(\sigma\chi)$ have uniform asymptotic expansions for large orders associated with them from which we can write the approximations [6]

$$I_{\sigma}(\sigma\chi)\approx\frac{1}{\sqrt{2\pi\sigma}}\frac{e^{\sigma\eta}}{(1+\chi^2)^{1/4}} \tag{33a}$$

and

$$K_{\sigma}(\sigma\chi)\approx\sqrt{\frac{\pi}{2\sigma}}\frac{e^{-\sigma\eta}}{(1+\chi^2)^{1/4}}, \tag{33b}$$

where

$$\eta=\sqrt{1+\chi^2}+\ln\left[\frac{\chi}{1+\sqrt{1+\chi^2}}\right]. \tag{33c}$$

Now, assuming that $\nu=\beta=0$ as in Case II, the potential is given by Eqs. (24) and (25). In this case, the function $J_{\mu+i\bar{p}\alpha}(\alpha\rho)$ (and all its variants) is complex when $\bar{p}$, α, μ, and ρ are all positive and real. Both its real and imaginary parts oscillate without bound as α increases when $\bar{p}$ and ρ are fixed. Thus, $J_{\mu+i\bar{p}\alpha}(\alpha\rho)$ should be used only near $\rho=0$ where it is finite, i.e., for interior types of boundary-value problems. For exterior boundary-value problems, where ρ can be large, it is necessary to use $H^{(2)}_{\mu+i\bar{p}\alpha}(\alpha\rho)$, the Hankel function of the second kind. It, too, is a complex function, but its real and imaginary parts are well behaved as its argument becomes large, i.e., its real and imaginary parts both oscillate but the oscillations decrease in magnitude toward zero as the argument increases. This same general behavior is exhibited by $H^{(1)}_{\mu-i\bar{p}\alpha}(\alpha\rho)$. Thus, for solving boundary-value problems, Eq. (24) may be replaced by

$$\Phi=[c_1J_{\mu+i\bar{p}\alpha}(\alpha\rho)+c_2H^{(2)}_{\mu+i\bar{p}\alpha}(\alpha\rho)]\times[c_3e^{i\mu\phi}e^{\alpha\zeta}+c_4e^{-i\mu\phi}e^{-\alpha\zeta}], \tag{34}$$

while Eq. (25) may be replaced by

$$\Phi=[c_1'J_{\mu-i\bar{p}\alpha}(\alpha\rho)+c_2'H^{(1)}_{\mu-i\bar{p}\alpha}(\alpha\rho)]\times[c_3'e^{-i\mu\phi}e^{\alpha\zeta}+c_4'e^{i\mu\phi}e^{-\alpha\zeta}]. \tag{35}$$

IV. RIGHT-HANDED AND LEFT-HANDED HELICAL HARMONICS

Obviously from consideration of Secs. I and II, one could assume either a right-handed helical coordinate system ($\bar{p}>0$) or a left-handed one ($\bar{p}<0$). Standard practice is to choose a right-handed system. However, the helix is a structure which can be either left-handed or right-handed, and this handedness is an intrinsic quality. A left-handed helix cannot be transformed into a right-handed helix and vice versa. This is completely unlike a sphere or a cylinder that have no such property. Thus, if a right-handed coordinate system is chosen, it is necessary to know how to deal with both right- and left-handed solutions in such a system. Using previous conventions [1], and assuming Case I where $P(\phi)=e^{-i\mu\phi}$ and $Z(\zeta)=e^{-i\beta\zeta}$,

$$\Phi_R=[a_1I_{(\mu-\bar{p}\beta)}(\beta\rho)+b_1I_{-(\mu-\bar{p}\beta)}(\beta\rho)]e^{-i\mu\phi}e^{-i\beta\zeta} \tag{36}$$

is considered to be a right-handed helical harmonic solution of Laplace's equation. The parameters and coordinates, β, ζ, μ, ϕ, ρ, and $\bar{p}$ are all assumed to be positive and real.

By assuming that $P(\phi)=e^{i\mu\phi}$, $Z(\zeta)=e^{-i\beta\zeta}$,

$$\Phi_L=[a_2I_{(\mu+\bar{p}\beta)}(\beta\rho)+b_2I_{-(\mu+\bar{p}\beta)}(\beta\rho)]e^{i\mu\phi}e^{-i\beta\zeta} \tag{37}$$

is considered to be a left-handed helical harmonic since it rotates along ϕ in the opposite direction to the solution in Eq. (36) but with the same dependence along ζ.

Assuming that $b_1=b_2=0$, by adding Eqs. (36) and (37), we get a combined right- and left-handed solution given by

$$\Phi=[a_1I_{(\mu-\bar{p}\beta)}(\beta\rho)e^{-i\mu\phi}+a_2I_{(\mu+\bar{p}\beta)}(\beta\rho)e^{i\mu\phi}]e^{-i\beta\zeta}. \tag{38}$$

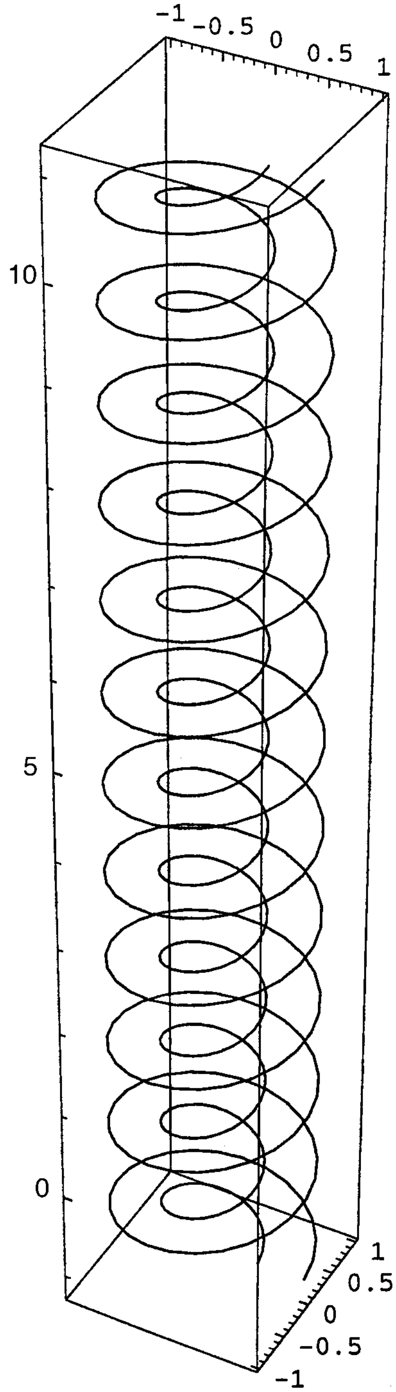

FIG. 2. Two nested helices with identical length and pitch but different radii set to different potential values ($a=0.5$, $b=1.0$, $p=1.0$)

V. A HELICAL BOUNDARY VALUE PROBLEM

Assume that two nested helices with identical pitches and lengths but different radii are positioned along the z axis as in Fig. 2. The inner helix with radius a has a potential, V_1. The outer helix with radius b is at a different potential, V_0. We wish to obtain the potential at all points in space. It is assumed also that the helices run from 0 to $+L$ along the z axis and that their length is much greater than their radii so that end effects may be neglected.

There are three regions where the potential must be considered: Region 1 where $0<\rho\leqslant a$, Region 2 where $a\leqslant\rho\leqslant b$, and Region 3 where $\rho\geqslant b$. In Region 1, the potential is assumed to have the general form,

$$\Phi_1=A[J_{i\bar{p}\alpha}(\alpha\rho)+J_{-i\bar{p}\alpha}(\alpha\rho)]\sinh\alpha\zeta. \tag{39a}$$

Φ_1 is assumed to be independent of the coordinate $\phi(\mu=0)$ and the form in Eq. (39a) is used to ensure that Φ_1 will be purely real since V_0 and V_1 are assumed to be real constants.

In Region 2, the potential is assumed to have the form

$$\Phi_2=\{B[J_{i\bar{p}\alpha}(\alpha\rho)+J_{-i\bar{p}\alpha}(\alpha\rho)] + C[H^{(2)}_{i\bar{p}\alpha}(\alpha\rho)+H^{(1)}_{-i\bar{p}\alpha}]\}\sinh\alpha\zeta. \tag{39b}$$

Finally in Region 3,

$$\Phi_3=D[H^{(2)}_{i\bar{p}\alpha}(\alpha\rho)+H^{(1)}_{-i\bar{p}\alpha}(\alpha\rho)]\sinh\alpha\zeta. \tag{39c}$$

Four boundary conditions must be met:

$$\Phi_1=V_1 \quad \text{on} \quad \rho=a, \quad \zeta=\zeta_0, \tag{40a}$$

$$\Phi_2=V_1 \quad \text{on} \quad \rho=a, \quad \zeta=\zeta_0, \tag{40b}$$

$$\Phi_2=V_0 \quad \text{on} \quad \rho=b, \quad \zeta=\zeta_0, \tag{40c}$$

$$\Phi_3=V_0 \quad \text{on} \quad \rho=b, \quad \zeta=\zeta_0. \tag{40d}$$

(Note that two conditions must be met simultaneously, i.e., $\rho=$constant and $\zeta=$constant to ensure that one is on a given helix.)

Applying the boundary conditions in Eq. (40) to the potentials in Eq. (39) and solving for the constants, *A*, *B*, *C*, and *D*, the following formulas are obtained:

$$A=\frac{V_1}{Ja\sinh\alpha\zeta_0}, \tag{41a}$$

$$B=\frac{V_1Hb-V_0Ha}{[JaHb-JbHa]\sinh\alpha\zeta_0}, \tag{41b}$$

$$C=\frac{-[V_1Jb-V_0Ja]}{[JaHb-JbHa]\sinh\alpha\zeta_0}, \tag{41c}$$

and

$$D=\frac{V_0}{Hb\sinh\alpha\zeta_0}, \tag{41d}$$

where

$$Ja=J_{i\bar{p}\alpha}(\alpha a)+J_{-i\bar{p}\alpha}(\alpha a), \tag{42a}$$

$$Jb=J_{i\bar{p}\alpha}(\alpha b)+J_{-i\bar{p}\alpha}(\alpha b), \tag{42b}$$

$$Ha=H^{(2)}_{i\bar{p}\alpha}(\alpha a)+H^{(1)}_{-i\bar{p}\alpha}(\alpha a), \tag{42c}$$

$$Hb=H^{(2)}_{i\bar{p}\alpha}(\alpha b)+H^{(1)}_{-i\bar{p}\alpha}(\alpha b). \tag{42d}$$

By substituting Eq. (41) back into Eq. (39), the potential from the two helices at any point in space is given by

$$\Phi_1 = \frac{V_1[J_{i\bar{p}\alpha}(\alpha\rho) + J_{-i\bar{p}\alpha}(\alpha\rho)]\sinh\alpha\zeta}{Ja\,\sinh\alpha\zeta_0} \tag{43a}$$

$$\begin{aligned}\Phi_2 = \{ & (V_1Hb - V_0Ha)[J_{i\bar{p}\alpha}(\alpha\rho) + J_{-i\bar{p}\alpha}(\alpha\rho)] \\ & -(V_1Jb - V_0Ja)[H^{(2)}_{i\bar{p}\alpha}(\alpha\rho) + H^{(1)}_{-i\bar{p}\alpha}(\alpha\rho)]\} \\ & \times \frac{\sinh\alpha\zeta}{[Ja\,Hb - Jb\,Ha]\sinh\alpha\zeta_0}\end{aligned} \tag{43b}$$

and

$$\Phi_3 = \frac{V_0[H^{(2)}_{i\bar{p}\alpha}(\alpha\rho) + H^{(1)}_{-i\bar{p}\alpha}(\alpha\rho)]\sinh\alpha\zeta}{Hb\,\sinh\alpha\zeta_0}. \tag{43c}$$

By considering the special case where $V_1 = 0$, i.e., the inner helix at $\rho = a$, $\zeta = \zeta_0$, and $\Phi_1 = \Phi_2 = 0$ there, the potential of Region 1 must be written as

$$\Phi_1 = [J_{i\bar{p}\alpha}(\alpha\rho) + J_{-i\bar{p}\alpha}(\alpha\rho)]\frac{\sinh\alpha\zeta}{\sinh\alpha\zeta_0}, \tag{44}$$

and to satisfy $\Phi_1 = 0$ on $\rho = a$, $\zeta = \zeta_0$, we must have the condition

$$[J_{i\bar{p}\alpha}(\alpha a) + J_{-i\bar{p}\alpha}(\alpha a)] = 2\,\mathrm{Re}[J_{i\bar{p}\alpha}(\alpha a)] = 0. \tag{45}$$

This condition is satisfied when α is chosen to be a zero of Eq. (45), α^*, causing

$$\mathrm{Re}[J_{i\bar{p}\alpha^*}(\alpha^* a)] = 0. \tag{46}$$

Thus, this special case where $V_1 = 0$ justifies our initial use of the $J_{i\bar{p}\alpha}(\alpha\rho)$ and $H^{(2)}_{i\bar{p}\alpha}(\alpha\rho)$ functions as opposed to the modified helical Bessel functions in Case I of Sec. II, since the modified helical Bessel functions have no real zeros associated with them.

VI. CONCLUSIONS

Helical harmonic solutions of Laplace's equation have been derived using a right-handed nonorthogonal helical coordinate system and a modified separation of variables technique. Unlike the Cartesian or cylindrical coordinate systems, in the helical system, two different sets of solutions are admitted, one right handed and one left handed. Consequently, the helical harmonics are used to solve the boundary-value problem of two nested helices with different radii set to different potential values.

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